An allocation method for indivisible goods: The generalized Hamilton Method

Martin Kohl

Leipzig University

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Motivation

- If indivisible goods have prices, side payments may eliminate the problems of indivisibility
- Research topic: how to solve allocation problems for goods without prices (most public goods, parliament seats,...)

Consider the following problem:

- three parties (A,B,C) in a parliament bargain over six seats in the budget committee
- let the claims be constant:

$$v(A) = 3, v(B) = v(C) = 1.$$

- Clearly, solution is $(3, 1, 1)$.
- but how to change the solution if the claims depend on the bargaining partners, the solution may be more difficult

$$\begin{aligned} v(A) &= 3, v(B) = v(C) = 1, \\ v(A, B) &= v(A, C) = 4, v(B, C) = 3, \\ v(A, B, C) &= 6. \end{aligned}$$

- What would be a good allocation of the 6 seats in the budget committee?
  - Shapley Value $(3, 1.5, 1.5)$ is not feasible
  - Contribution of A is constant, thus payoff of A should be 3?
  - Equal division for B and C, i.e. again $(3, 1, 1)$? One seat remaining...
- How about choosing the vectors which are closest to the Shapley Value?

Outline

- Literature
- Games with transferable utility and the Shapley Value
- Apportionment problems and the Hamilton Method
- TU-games + Apportionment problems = Integer games
- Axiomatization of the Hamilton-Shapley-Method
- Criticism of the Hamilton-Shapley-Method
- Outlook: The Webster-Shapley-Method or Avoiding the Paradoxes

Literature

Games with transferable utility

- Shapley, L. (1953). A value for n-person game
- van den Brink, R. (2007). Null or nullifying players: the difference between the shapley value and equal division solutions

Overviews on Apportionment problems

- Balinski, M.L.,Young, P. (1974). A New Method for Congressional Apportionment
- Sainte-Laguë, A. (1910) La représentation proportionnelle et la méthode des moindres carrés

The example from the introduction is close connected to two branches of economic theory.

i) **Games with transferable utility:**
Every subset S of finite player set N generates cooperation benefits $v(S) \in \mathbb{R}$.
Solutions of games with transferable utility decide how to distribute the worth generated by all the players.
Solution vectors are real-valued.

Definition

The Shapley Value of a game with transferable utility (N, v) is given by

$$Sh_i(N, v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!\,(|N| - |S| - 1)!}{|N|!} \left(v(S \cup i) - v(S)\right).$$

ii) **Apportionment problems:**
Every player i has a claim c_i on a number of seats E within a parliament.
Solutions of Apportionment problems decide how to allocate E seats.
Solution vectors are integer-valued.

Definition (Hamilton (1792)/ Hare-Niemeyer Method (1970))

For any apportionment problem $\left(N, (c_i)_{i \in N}, E\right)$ the Hamilton solution is derived in two steps

1. Every agent obtains the integer of his/her proportional claim, i.e. agent i obtains

$$\left\lfloor \frac{c_i}{\sum_{j \in N} c_j} \cdot E \right\rfloor$$

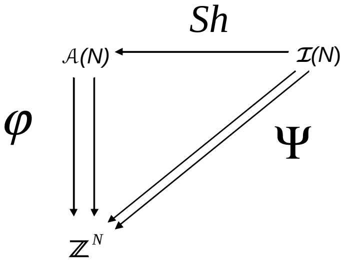
2. if x seats are still not allocated, the x agents whose fractional remainders are the largest, obtain an additional seat. If there are ties, the solution may consist of more than one payoff vector.

The combination of the two branches yields a new set of problems within the normative economic theory.

Definition (Integer games)

An Integer game is a game with transferable utility with $v(N) \in \mathbb{Z}$. Solutions may consist of several vectors and these vectors are integer-valued. For a fixed player set, the set of all integer games is denoted by $\mathcal{I}(N)$

The example from the Introduction is an integer game.
We try to find a solution Ψ such that the following diagram commutates:



Axioms for Integer games I

We search for a solution for integer games Ψ, which may or may not fulfill the following axioms:

Definition (Efficiency axiom)

A solution $\Psi : \mathcal{I}(N) \rightrightarrows \mathbb{Z}^N$ fulfills the Efficiency axiom iff for any $v \in \mathcal{I}(N)$ and any $\varphi \in \Psi(N, v)$ we have $\sum_{i \in N} \varphi_i = v(N)$.

The cooperation benefit of all players has to be allocated.

Definition (Dominating)

A player i dominates another player j (in (N, v)), iff $v(S \cup \{i\}) \geq v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i,j\}$ and $v(S \cup \{i\}) > v(S \cup \{j\})$ for at least one coalition $S \subseteq N \setminus \{i,j\}$.

Definition (Dominance axiom)

A solution $\Psi : \mathcal{I}(N) \rightrightarrows \mathbb{Z}^N$ fulfills the Dominance axiom iff for any player $i \in N$ dominating player $j \in N$, we have $\varphi_i \geq \varphi_j$ for all $\varphi \in \Psi(N, v)$.

This ensures, that less productive players never obtain a higher payoff than more productive ones.

Definition (Symmetric Players)

Players i, j are called symmetric (in (N, v)) iff for all $S \subseteq N \backslash \{i, j\} : v(S \cup \{i\}) = v(S \cup \{j\})$.

Definition (Symmetry axiom)

A solution $\Psi : \mathcal{I}(N) \rightrightarrows \mathbb{Z}^N$ fulfills the Symmetry axiom iff for any symmetric players $i, j \in N$ and any $\varphi \in \Psi(N, v)$, the vector τ given by

$$\tau_k = \begin{cases} \varphi_i, & k = j \\ \varphi_j, & k = i \\ \varphi_k, & \text{otherwise} \end{cases}$$

lies in $\Psi(N, v)$.

If the payoffs of two equally productive players differ for an element of $\Psi(N, v)$, the vector derived by switching their payoffs should also lie within $\Psi(N, v)$.

Definition (Dummy Player axiom)

A solution $\Psi : \mathcal{I}(N) \rightrightarrows \mathbb{Z}^N$ fulfills the Dummy axiom, if for any integer game $v \in \mathcal{I}(N)$, any player $i \in N$ with

$$v\left(S \cup \{i\}\right) - v\left(S\right) = v\left(T \cup \{i\}\right) - v\left(T\right)$$

for all $S, T \subseteq N \setminus \{i\}$ and any $\varphi \in \Psi\left(N, v\right)$, we have

$$\lfloor v\left(\{i\}\right) \rfloor \leq \varphi_i \leq \lceil v\left(\{i\}\right) \rceil .$$

A player whose contribution is constant should obtain the integer part of this contribution or the smallest integer not larger than its contribution, thus a player with a constant contribution of 3.5 should obtain either 3 or 4.

Definition (Fairness axiom)

A solution $\Psi : \mathcal{I}(N) \rightrightarrows \mathbb{Z}^N$ fulfills the Fairness axiom iff for any set $S \subseteq N$ and any integer games $v, w \in \mathcal{I}(N)$ given by

$$w(T) = \left\{ \begin{array}{cc} v(T), & S \nsubseteq T \\ v(T) + |S|, & S \subseteq T \end{array} \right\},$$

$\tau \in \Psi(N, v)$ if and only if the vector φ

$$\varphi_k = \left\{ \begin{array}{cc} \tau_k + 1, & k \in S \\ \tau_k, & k \notin S \end{array} \right.$$

lies within $\Psi(N, w)$.

If the productivity of a subset of players increases, while the productivity of all other players stays the same, only the players whose productivity increased should benefit of the larger pie.

Definition (Shapley Approximation Axiom)

A solution $\Psi : \mathcal{I}(N) \rightrightarrows \mathbb{Z}^N$ fulfills the Shapley-Approximation axiom, iff for any $v, w \in \mathcal{I}(N)$ with $Sh\,(N, v) = Sh\,(N, w)$, we have

$$\Psi\,(N, v) = \Psi\,(N, w)\,.$$

This axiom reflects that the claims of the players are represented by the Shapley Value.

The Hamilton-Shapley Method I

Theorem

There is a unique solution for integer games that fulfills

- *the Efficiency axiom,*
- *the Dominance axiom,*
- *the Symmetry axiom,*
- *the Dummy Player axiom,*
- *the Fairness axiom, and*
- *the Shapley Approximation axiom.*
  This solution is given by

$$HS\left(N, v\right) = Ham\left(N, Sh\left(N, v\right), v\left(N\right)\right)$$

and is called the Hamilton-Shapley method.

Thus, we first give every agent the integer part of its Shapley payoff and then give an additional seat to those players whose remainder is the largest.

Example

Consider the voting results

$$\begin{aligned} \text{party } 1 &: \quad 30.000 \text{ votes,} \\ \text{party } 2 &: \quad 30.000 \text{ votes,} \\ \text{party } 3 &: \quad 10.000 \text{ votes.} \end{aligned}$$

The house size is either 10 or 11. The resulting integer games are

$$\begin{aligned} v(1) &= v(2) = \frac{30}{7}, v(3) = \frac{10}{7} \text{ and } v(S) = \sum_{i \in S} v(i) \\ w(1) &= w(2) = \frac{33}{7}, w(3) = \frac{11}{7}, \text{ and } w(S) = \sum_{i \in S} w(i) \end{aligned}$$

We obtain

$$Ham(Sh(N,v),10) = (4,4,2), \text{ but } Ham(Sh(N,w),11) = (5,5,1).$$

Same votes, but larger house implies less seats?!

Criticism of the Hamilton-Shapley Method II

The Population Paradox

Consider the census' of the US of 1900 and 1910 and the associated seats in the House of Representatives according to the Hamilton-Shapley method.

	1900		1910	average Growth in Pop.	
	Population	Seats	Population	per year in %	Seats
Virginia	1,854,185	10	2,061,612	+1.07	9
Maine	694,466	3	742,371	+0.67	4
US	74,562,608	386	91,072,117	+2.02	386

Even though the claim of Virginia is stronger in 1910 than in 1900 and the House size stays the same, a seat is transferred from Virginia to Maine.

Definition (Webster Method)

Let N be the finite player set and $(c_i)_{i\in N}$ be the claims of the players and E the house size. The Webster solution is defined by

$$\begin{aligned} Web_i(c, E) &= \left[\frac{E}{\lambda} \cdot c_i\right], \\ \text{where } \lambda \text{ is such that } \sum_{i \in N} \left[\frac{E}{\lambda} \cdot c_i\right] &= E. \end{aligned}$$

$[x]$ is the usual rounding method.

The Webster method was proposed by Daniel Webster in 1830, but it is only since 1880 that is has been used in the allocation of the House of Representatives. In Europe it was introduced by André Saint-Laguë in 1910.

Definition

Let $v \in \mathcal{I}(N)$ be an integer game. The Webster-Shapley Method is defined by

$$WS(N, v) = Web(Sh(N, v), v(N)).$$

Theorem

The Webster-Shapley Method fulfills the following axioms

- *Efficiency,*
- *Symmetry,*
- *Shapley Approximation,*
- *avoids Alabama Paradox and Population Paradox*

but does not fulfill

- *Dummy player*
- *Fairness*

Summary

- Integer games describe situations, that are much more general than apportionment problems and reflect more complicated bargaining situations
- A first solution for integer games is established, which extends the Hamilton method of apportionment problems
- If we agree on the six axioms presented, we have to choose the Hamilton-Shapley Method
- Nevertheless, the solution suffers from the Alabama Paradox and the Population Paradox
- Another solution, the Webster-Shapley Method, is presented, and does not suffer from the these paradoxes, but needs to be axiomatized (work in progress)

Thank you!